TURBULENT MIXING IN A SYSTEM OF PLANE NONISOTHERMAL JETS

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An examination is made of turbulent flow in the main section in an infinite system of plane nonisothermal jets. The results of calculation are compared with experimental data.

We shall examine the flow formed in the basic section of the mixing zone of an infinite system of plane nonisothermal jets, discharging from nozzles of width $\lambda/2$ each. The jet exits we shall assume to be such (Fig. 1) that periodic flow occurs with period λ. In this case the streamlines passing through the middle of the nozzles will be straight lines parallel to the axis ox, and therefore it will be sufficient to examine the flow between any two streamlines separated by a distance λ, for example between lines ac and bd.

The analogous problem for the isothermal case was examined in [1].

We shall assume that $c_pT \gg u^2/2$, $\mathrm{Pr}_T = 1$ and $c_p =$ = const, and then the basic equations describing turbulent motion in the mixing zone will take the form

$$\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\rho\varepsilon\frac{\partial u}{\partial y}\right),$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0, \qquad (1)$$

$$\rho u\frac{\partial T}{\partial x} + \rho v\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left(\rho\varepsilon\frac{\partial T}{\partial y}\right),$$

$$p = \rho RT. \qquad (1')$$

We shall write the boundary conditions in the form

$$\frac{\partial u}{\partial y} = \frac{\partial T}{\partial y} = v = 0 \ \text{ for } y = 0,\ \pm\frac{\lambda}{2}; \qquad (2)$$

$$u = u(y),\quad T = T(y),\quad p = p_i \ \text{ for } x = x_i. \qquad (2')$$

Transforming the first and third equations of system (1) with the help of the continuity equation, and integrating them with respect to y in the range $[-\lambda/2, \lambda/2]$, subject to conditions (2), we obtain the following integral conditions for conservation of momentum and heat:

$$\int_{-\lambda/2}^{\lambda/2}(p + \rho u^2)\,dy = I_0 = \text{const}, \qquad (3)$$

$$\int_{-\lambda/2}^{\lambda/2}\rho uT dy = H_0 = \text{const}. \qquad (4)$$

Integrating the continuity equation over the same limits, we find an integral condition for conservation of mass:

$$\int_{-\lambda/2}^{\lambda/2}\rho u dy = M_0 = \text{const}. \qquad (5)$$

At infinity (for $x \to \infty$), because the mixing is complete, the stream will be uniform, i. e., it will move with constant velocity u_∞, temperature T_∞, pressure p_∞, and density ρ_∞.

For the turbulent viscosity ε we shall make use of the Prandtl hypothesis, according to which

$$\varepsilon = K\lambda(u_{\max} - u_{\min}) = K\lambda\left[u(0) - u(\lambda/2)\right]. \qquad (6)$$

Following substitution of (6) into (1), and going over to dimensionless variables according to the formulas $x = x'\lambda$, $y = y'\lambda$, $u = u'u_\infty$, $v = v'u_\infty$, $T = T'T_\infty$, $p = p'\rho_\infty u_\infty^2$, and $\rho = \rho'\rho_\infty$, system (1) may rewritten in the form (for convenience the primes in the dimensionless variables are omitted)

$$\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + K\left[u(0) - u\left(\frac{1}{2}\right)\right]\frac{\partial}{\partial y}\left(\rho\frac{\partial u}{\partial y}\right),$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0,$$

$$\rho u\frac{\partial T}{\partial x} + \rho v\frac{\partial T}{\partial y} = K\left[u(0) - u\left(\frac{1}{2}\right)\right]\frac{\partial}{\partial y}\left(\rho\frac{\partial T}{\partial y}\right),$$

$$p = b_0\rho T, \qquad (7)$$

where

$$b_0 = RT_\infty/u_\infty^2.$$

The dimensionless boundary conditions coincide in form with (2) and (2').

We shall seek expressions for u, T, and p in the form of the expansions

$$u = 1 + \frac{a_1(y)}{x} + \frac{a_2(y)}{x^2} + \frac{a_3(y)}{x^3} + \ldots,$$

$$T = 1 + \frac{c_1(y)}{x} + \frac{c_2(y)}{x^2} + \frac{c_3(y)}{x^3} + \ldots,$$

$$p = b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \frac{b_3}{x^3} + \ldots. \qquad (8)$$

Eliminating v by means of the continuity equation, transformed to the form

$$v = -\frac{1}{\rho}\int_0^y \frac{\partial}{\partial x}(\rho u)\,dy, \qquad (9)$$

and the density ρ by means of the equation of state, from the first and third equations of systems (7), substituting the expansion (8) into these equations and equating coefficients of the same powers of x on the left and right sides, we obtain the following system of equations to determine the coefficients a_i and c_i of the expansion

$$a_1'' + a'/K\delta_1 = -b_1/K\delta_1,$$

$$a_2'' + 2a_2/K\delta_1 = a_1'c_1' - a_1''c_1 - [b_1/b_0 + \delta_2/\delta_1]a_1'' - 2a_1c_1/K\delta_1 -$$

$$-\Lambda_1 a_1/Kb_0\delta_1 + a_1'\int_0^y \Lambda_1\,dy/Kb_0\delta_1 - (2c_1b_1 + 2b_2)/K\delta_1,$$

$$a_3'' + 3a_3/K\delta_1 = c_1'a_2' + a_1'c_2' + [b_1/b_0 + \delta_2/\delta_1]a_1'c_1' -$$

$$-a_1''c_2 - a_2''c_1 - [b_1/b_0 + \delta_2/\delta_1][a_1''c_1 + a_2''] -$$

$$-\left[\frac{\delta_3}{\delta_1} + \frac{b_1}{b_0}\,\frac{\delta_2}{\delta_1} + \frac{b_2}{b_0}\right]a_1'' - \frac{4a_2c_1 + 2a_1c_2 + a_1c_1^2}{K\delta_1} -$$

$$-\left(\Lambda_1[2a_2 + 2a_1c_1] - \Lambda_2 a_1 - 2a_1'\int_0^y \Lambda_2\,dy\right)/Kb_0\delta_1 +$$

$$+[2a_1'c_1 + a_2']\int_0^y \Lambda_1\,dy/Kb_0\delta_1 -$$

$$-(3b_3 + 4b_2c_1 + 2b_1c_2 + b_1c_1^2)/K\delta_1;$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \quad (10)$$

$$c_1'' + c_1/K\delta_1 = 0,$$

$$c_2'' + 2c_2/K\delta_1 = c_1'^2 - c_1c_1'' - \left[\frac{b_1}{b_0} + \frac{\delta_2}{\delta_1}\right]c_1'' -$$

$$-\left(2b_0c_1^2 + c_1\Lambda_1 - c_1'\int_0^y \Lambda_1\,dy\right)/Kb_0\delta_1,$$

$$c_3'' + 3c_3/K\delta_1 = 2c_1'c_2' + [b_1/b_0 + \delta_2/\delta_1]c_1'^2 - c_1c_2'' - c_2c_1'' -$$

$$-\left[\frac{b_1}{b_0} + \frac{\delta_2}{\delta_1}\right][c_1c_1'' + c_2''] - \left[\frac{\delta_3}{\delta_1} + \frac{b_1}{b_0}\,\frac{\delta_2}{\delta_1} + \frac{b_2}{b_0}\right]c_1'' -$$

$$-\left(6c_1c_2 + c_1^3 + \Lambda_1[2c_1^2 + 2c_2] + c_1\Lambda_2\right)/Kb_0\delta_1 +$$

$$+\left([c_2' + 2c_1c_1']\int_0^y \Lambda_1\,dy + 2c_1'\int_0^y \Lambda_2\,dy\right)/Kb_0\delta_1;$$

$$\cdots\cdots\cdots\cdots\cdots\cdots, \quad (11)$$

where we have introduced the notation

$$\delta_i = a_i(0) - a_i(1/2) \quad (i = 1,\ 2,\ 3),$$

$$\Lambda_1 = b_1 - b_0c_1 + b_0a;$$

$$\Lambda_2 = b_2 - b_0c_2 - c_1b_1 + b_0c_1^2 + a_1b_1 - b_0a_1c_1 + b_0a_2;$$

$$\Lambda_3 = b_3 - b_0c_3 - c_2b_1 + 2b_0c_1c_2 - c_1b_2 + c_1^2b_1 - b_0c_1^3 +$$

$$+ a_1b_2 - a_1b_0c_2 - a_1c_1b_1 + a_1b_0c_1^2 + a_2b_1 - a_2b_0c_1 + a_3b_0.$$

The boundary conditions for the system obtained are

$$\frac{\partial a_i}{\partial y} = \frac{\partial c_i}{\partial y} = 0 \quad \text{for } y = 0,\ \pm\frac{1}{2}. \quad (12)$$

Substitution of the expansion (9) into the integral conditions (3)–(5), written for dimensionless variables, and equating the coefficients for the same powers of x on the left and right sides leads to the relations

$$I_0 = \rho_\infty u_\infty^2\lambda(b_0 + 1),$$

$$\int_{-1/2}^{1/2}(a_1b_0 + \Lambda_1 + b_0b_1)\,dy = 0,$$

$$\int_{-1/2}^{1/2}(a_2b_0 + a_1\Lambda_1 + \Lambda_2 + b_0b_2)\,dy = 0,$$

$$\int_{-1/2}^{1/2}(a_3b_0 + b_2\Lambda_1 + a_1\Lambda_2 + \Lambda_3 + b_0b_3)\,dy = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \quad (13)$$

$$H_0 = \rho_\infty u_\infty T_\infty\lambda,$$

$$\int_{-1/2}^{1/2}(b_0c_1 + \Lambda_1)\,dy = 0,$$

$$\int_{-1/2}^{1/2}(b_0c_2 + c_1\Lambda_1 + \Lambda_2)\,dy = 0,$$

$$\int_{-1/2}^{1/2}(b_0c_3 + c_2\Lambda_1 + c_1\Lambda_2 + \Lambda_3)\,dy = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \quad (14)$$

$$M_0 = \rho_\infty u_\infty\lambda,$$

$$\int_{-1/2}^{1/2}\Lambda_1\,dy = 0,$$

$$\int_{-1/2}^{1/2}\Lambda_2\,dy = 0$$

$$\int_{-1/2}^{1/2}\Lambda_3\,dy = 0$$

$$\cdots\cdots\cdots \quad (15)$$

Then system (10) and (11) together with conditions (13) represents a closed system of ordinary linear differential equations, which may be integrated successively.

The general solution of the first equation of system (10) has the form

$$a_1 = A_1\cos\sqrt{1/K\delta_1}\;y + B_1\sin\sqrt{1/K\delta_1}\;y - b_1. \quad (16)$$

Because of conditions (12), we must put $B_1 = 0$,

$$\sqrt{1/K\delta_1} = 2\pi. \quad (17)$$

The solution of (16) may be rewritten in the form

$$a_1 = A_1\cos(2\pi y) - b_1. \quad (18)$$

The solution of the first equation of system (11), satisfying condition (12), is

$$c_1 = A_1'\cos(2\pi y) = c_1(0)\cos(2\pi y). \quad (19)$$

Satisfying the second condition of system (13), we obtain $b_1 = 0$. From equality (18), using (17), we find

$$A_1 = a_1(0) = -a_1(1/2) = 1/8\pi^2K. \quad (20)$$

The constant of integration $c_1(0)$ is as yet undefined.

In a similar way we may integrate the remaining equations and find that

$$u = 1 + \frac{1}{8\pi^2K}\left\{\frac{\cos(2\pi y)}{x} + \left[\frac{0.5\,l_1}{b_0 - 1} + C\cos(2\pi y) +\right.\right.$$

$$\left.+ 0.5\,c_1(0)\cos(4\pi y)\right]\Big/x^2 +$$

$$+\left[\frac{Cl_1}{b_0 - 1} + \left\langle l_1\left[\frac{3l_1 + b_0c_1(0)}{2(1 - b_0)}\right] +\right.\right.$$

$$\left.\left.+\frac{c_1(0)^2}{4} + c^2\right\rangle\cos(2\pi y)\right]\Big/x^3 + \left|c_1(0)\,C\cos(4\pi y) +\right.$$

$$\left.\left.+ 0.25\,c_1(0)^2\cos(6\pi y)\right|/x^3 + \ldots\right\}, \quad (21)$$

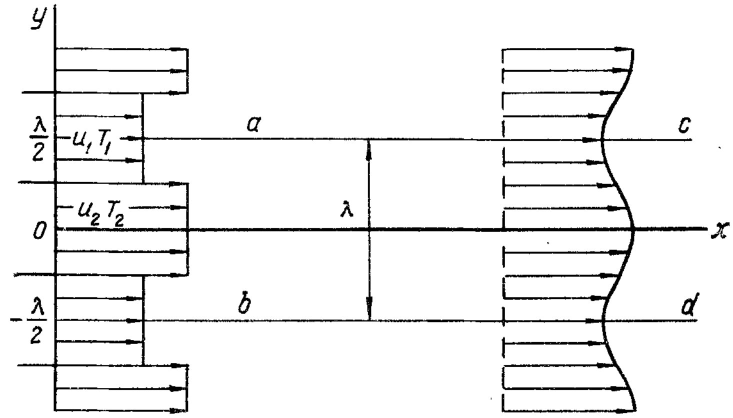

Fig. 1. Flow model.

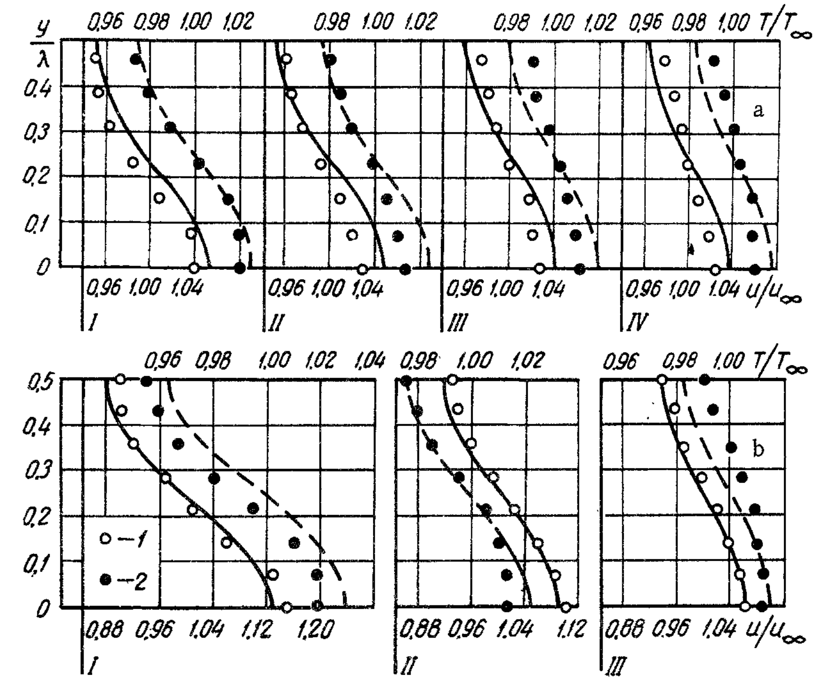

Fig. 2. The distribution of velocity (points 1—experiment, solid line is calculation) and of temperature (points 2—experiment, dashed line is calculation) in the cross sections I—x/λ = 6.55, II—9.62, III—12.7, IV—15.76 for u_1 = 54.6 m/sec, u_2 = 49.4 m/sec, T_1 = = 400° K, T_2 = 304° K (a), and in the cross sections I—x/λ = 6.075, II—8.93, III—11.78 for u_1 = 60.8 m/sec, u_2 = 46.2 m/sec, T_1 = 396° K, T_2 = 310° K (b).

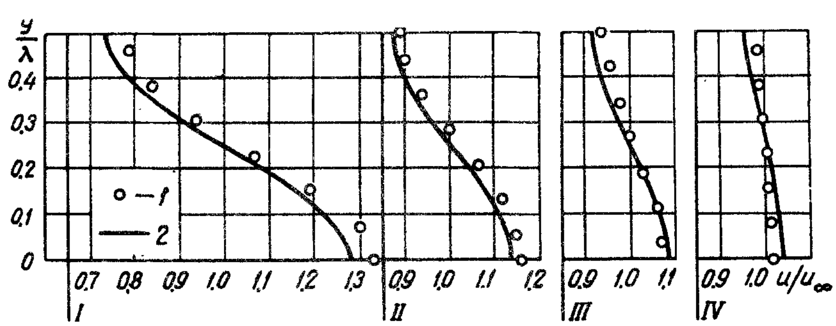

Fig. 3. Distribution of velocity in the cross sections I—x/λ = 3.46, II—6.55, III—9.62, IV—15.76 for u_1 = = 29.2 m/sec, u_2 = 48.75 m/sec; 1) experiment; 2) theory.

$$T = 1 + c_1(0) \times$$
$$\times \left\{ \frac{\cos(2\pi y)}{x} + \frac{-0.5\, l_1 + C\cos(2\pi y) + 0.5\, c_1(0)\cos(4\pi y)}{x^2} + \right.$$
$$+ \left[-Cl_1 + \left\langle l_1 \left[\frac{(3-b_0)\, l_1 + b_0 c_1(0)}{2(1-b_0)} \right] + \right.\right.$$
$$\left.\left. + \frac{c_1(0)^2}{4} + C^2 \right\rangle \cos(2\pi y) \right] \Big/ x^3 +$$
$$\left. + \frac{c_1(0)\, C\cos(4\pi y) + 0.25\, c_1(0)^2 \cos(6\pi y)}{x^3} + \ldots \right\}, \quad (22)$$

$$p = b_0 + \frac{b_0 l_1}{16\pi^2 K (1-b_0)\, x^2} \left\{ 1 + \frac{2C}{x} + \ldots \right\}. \quad (23)$$

Here $l_1 = a_1(0) - c_1(0) = 1/8\pi^2 K - c_1(0)$; $c_1(0)$ and C are constants of integration which are determined by conditions (2').

It may be seen from formulas (21)–(23) that the profiles of velocity and of temperature, as well as the variation of pressure along the axis ox, depend on the initial distributions of velocity and temperature, and on the initial value of pressure.

The values of the stream parameters at infinity are determined by simultaneous solution of the equations

$$I_0 = \rho_\infty u_\infty^2 \lambda (b_0 + 1),$$
$$H_0 = \rho_\infty u_\infty T_\infty \lambda,$$
$$M_0 = \rho_\infty u_\infty \lambda,$$
$$b_0 = RT_\infty / u_\infty^2, \quad (24)$$

where u_∞, T_∞, ρ_∞ and b_0 are unknowns.

In order to check the solution obtained we performed calculations of the profiles of velocity and temperature in the main part of a system of plane isothermal and nonisothermal jets. The values of profiles of velocity, temperature, and pressure in the initial section, and also the abscissa of the initial section were taken from experiments conducted in an experimental installation in the hydrodynamics laboratory of the LPI. The constants $c_1(0)$ and C were determined from the condition of best agreement between the analytical and the experimental profiles of velocity and temperature at the initial section of the main part, having substituted into (21) and (22) values of abscissa x equal to the values at the initial section x_i.

We note that the calculations were performed with the first three terms of the expansion of u and T in negative powers of x taken into account, while the number of constants in (21), (22), subject to determination, depends on the number of terms of the series in (8) taken into account.

In the case shown in Fig. 2, a, the dimensionless abscissa of the initial section is equal to $x_i/\lambda = 6.55$. The conditions at the entrance were: $u_2/u_1 = 0.9$; $T_1 = 400°$ K, $T_2 = 304°$ K, $u_\infty = 40.8$ m/sec, $T_\infty = 350°$ K. For $c_1(0)$ and C the following values were found:

$$c_1(0) = 0.4; \quad C = -3.8.$$

The results of calculation for other initial data ($x_i/\lambda = 6.075$, $u_\infty = 41.4$ m/sec, $T_\infty = 356°$ K; $u_2/u_1 = 0.76$, $T_1 = 396°$ K, $T_2 = 310°$ K) are shown in Fig. 2, b. In this case it turned out that

$$c_1(0) = 0.217; \; C = -0.359.$$

Figure 3 shows the development of the velocity profile in isothermal conditions with initial data $x_i/\lambda = 3.46$, $u_\infty = 30.5$ m/sec. The constant of integration C turned out to be 0.585.

A value K = 0.016 for the empirical turbulence constant was assumed in all cases.

In the cases examined the variation in pressure, both in experiment and in the calculations, proved to be negligibly small.

The results of comparison of the calculated curves with experiment permit us to conclude that there is satisfactory agreement between the calculated and the experimental data.

NOTATION

x, and y are the coordinate axes; u and v are the velocity components along the x and y axes, respectively; ρ is density; c_p is specific heat at constant pressure; T is absolute temperature; p is pressure; Pr_T and ε are, respectively, the turbulent analogs of Prandtl number and kinematic viscosity; R is the gas constant; x_i is the abscissa of the initial cross section of the main section; K is the empirical turbulence constant, determined from experiment.

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